Last time:

* Ampère's law for magnetized materials

$$\underline{H} \equiv \frac{1}{\mu_0}\underline{B} - \underline{M} \quad \text{auxiliary field}$$

$$\nabla \times \underline{H} = \underline{J}_f$$

$$\oint \underline{H} \cdot d\underline{l} = I_{f\,enc}$$

* Linear media

$\underline{M} = \chi_m \underline{H}$; where $\chi_m \equiv$ magnetic susceptibility

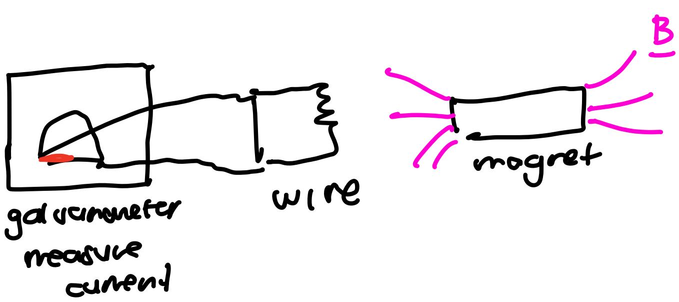
$\underline{B} = \mu \underline{H}$ where $\mu \equiv \mu_0(1+\chi_m)$ permeability

* Ferromagnets → spin alignment
  → Magnetization is history dependent
  → hysteresis.

Today: Electromagnetic induction and Faraday's law

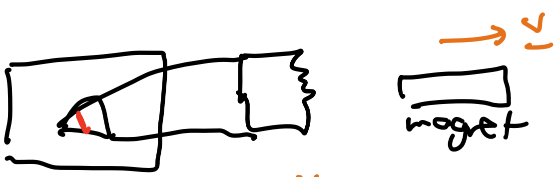
Faraday's experiments → Read Purcell for details

①



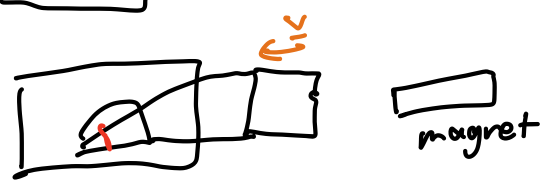
Both magnet & wire are at rest → zero current measured
nothing happens

②



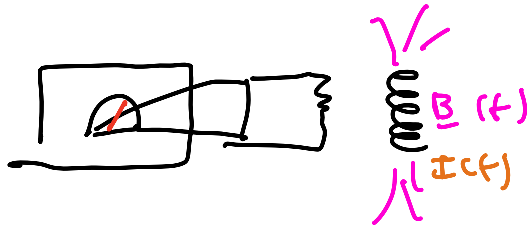
we move the magnet & we get a current in the wire

③



This also produces a current

④



A varying magnetic field produces a current in the wire.

What can we conclude?

* A constant magnetic field does NOT induce a current
* A changing magnetic field will act as an emf electromotive force and induce a current in the wire.

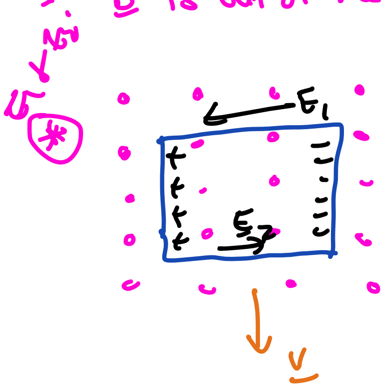
Why is there a current in the wire?

Problem 3:

Moving loop in a uniform magnetic field

• $\underline{B}$ is out of the page

$$\underline{F} = q(\underline{v} \times \underline{B})$$



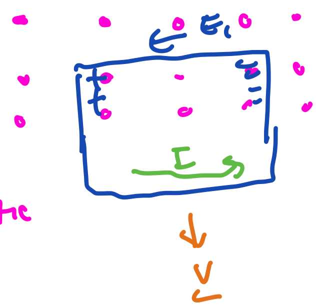
We move the loop $\perp$ to $\underline{B}$ $\Rightarrow$ there will be a force since $\underline{v} \times \underline{B} \neq 0$

$\Rightarrow e^-$ in the wire will experience $\underline{F}$ and move

$\underline{E}_1$ will be created due to the separation of charges due to the Lorentz force.

which then will generate $\underline{E}_2 = -\underline{E}_1$

What happens if $\underline{B}$ is non uniform?



electromagnetic induction

We move the loop at velocity $\underline{v}$

Above $B = B_0$ uniform

Below $B = 0$

Again Lorentz $\underline{F}$ will induce movement of charges $\underline{E}_1$ which results in separation of charges

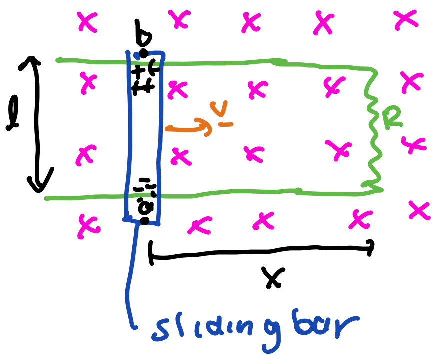
At the bottom there is no opposing $\underline{F}$ so charges flow $\Rightarrow I$ current

This is the end of electrostatics

$$\oint \underline{E} \cdot d\underline{l} \neq 0 \quad \text{or} \quad \nabla \times \underline{E} \neq 0$$

Motional emf

Consider a conducting bar sliding along 2 frictionless conducting rails, connected by a resistance R. The bar is moving through a uniform magnetic field, which points into the page.



Bar sliding along rails a distance l apart.

Rails connected by a resistance R

Separation between rails is ℓ

Bar is moving at velocity $\underline{v}$ in a uniform magnetic field

x is the distance from the bar edge to the edge of the loop

What happens as the bar moves?

Particles on the bar with a positive charge $(q>0)$ will experience a magnetic force:

$$\underline{F}_B = q\,\underline{v} \times \underline{B}$$

This force will push the + charges to the top of the bar

⇒ There is an accumulation of ⊕ charges at the top of the bar

⇒ There is an accumulation of ⊖ charges at the bottom

⇒ Separation of charges will give rise to an electric field $\underline{E}$ inside the bar, which will produce a downward electric force $\underline{F}_e$

$$\underline{F}_e = q\,\underline{E}$$

So now we have a potential difference between both ends of the bar:

$$\varphi_{ab} = -\int_a^b \underline{E}\cdot d\underline{s} = -\frac{1}{q}\int_a^b \underline{F}_e \cdot d\underline{s} \quad *$$

Recall we defined (lecture 12)

$$\mathcal{E} = \frac{dW}{dQ} \quad \text{work done / unit charge} \quad *$$

$$\Rightarrow \mathcal{E} = \frac{dW}{dQ} = -\frac{1}{q}\int \underline{F}_e \cdot d\underline{s} = \frac{1}{q}\int \underline{F}_B \cdot d\underline{s} = \int \underline{f}_B \cdot d\underline{s}$$

at equilibrium $\underline{F}_e = -\underline{F}_B$

define $\underline{f}_B = \dfrac{\underline{F}_B}{q}$

Since $\mathcal{E}$ arises from the motion of the wire, this potential difference is known as motional emf

$$\boxed{\mathcal{E} = \oint \underline{f}_B \cdot d\underline{s}}$$

Motional emf

In the case of the loop:

$$\varepsilon = \oint \underline{f}_B \cdot d\underline{s} = \frac{F_B}{q}\int ds = \frac{qvB}{q}\, l = Blv$$

← We'd like to write this in terms of the flux of $\underline{B}$

What is the flux of $\underline{B}$?

$$\boxed{\Phi_B = \int \underline{B} \cdot d\underline{a}}$$

magnetic flux through a loop

We have calculated

$$\varepsilon = Blv = \boxed{Bl\frac{dx}{dt}}$$ ←

On the other hand the magnetic flux in this case:

$$\Phi_B = \int \underline{B} \cdot d\underline{a} = B\int da = Blx \Rightarrow \frac{\partial \Phi_B}{\partial t} = \boxed{Bl\frac{dx}{dt}} = Bl(-v)$$

$\frac{dx}{dt}$ is negative because flux decreases as bar moves

$$\boxed{\varepsilon = -\frac{\partial \Phi_B}{\partial t}}$$

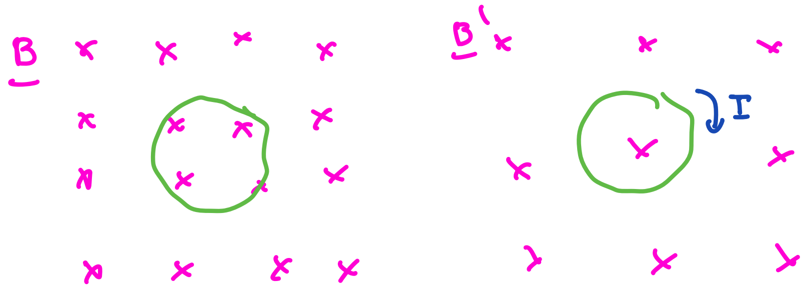
Flux rule for motional emf.

The minus sign indicates that the direction of the induced current will be such to oppose a change in flux of B

→ electromagnetic inertia

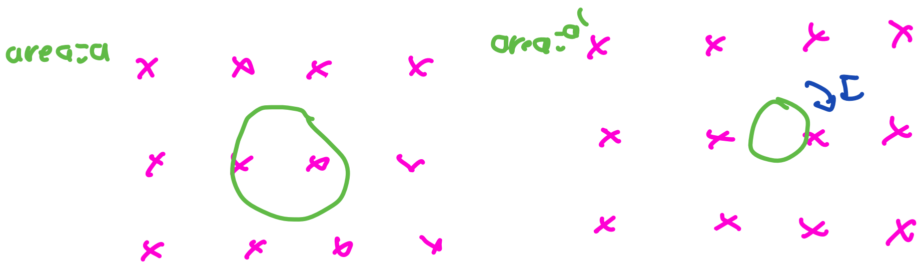
$\varepsilon = -\frac{d\Phi_B}{dt}$ holds in general for loops that change shape and area (see derivation in handwritten notes)

We can see that motional emf can be induced by:



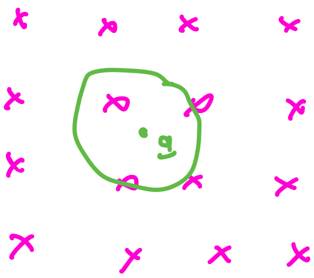
$\underline{B}' < \underline{B}$

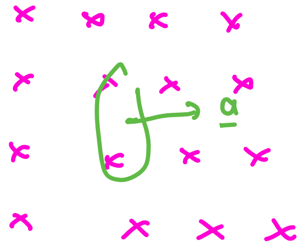
① Changing the magnitude of B with time



$a > a'$

② Changing the area of the loop

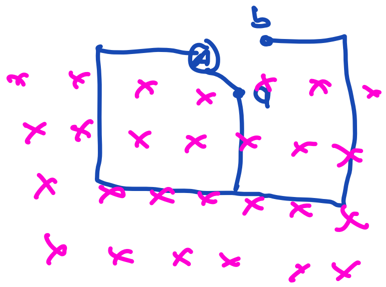




3) Changing the angle ~~between~~ between $\underline{B}$ and $\underline{a}$

Disclaimer:: Flux rule comes from Lorentz law
• Assumes you have a single loop

It won't work



when we move to switch from $a \to b$
we increase area $\Rightarrow$ twice as much flux
but no motional emf.

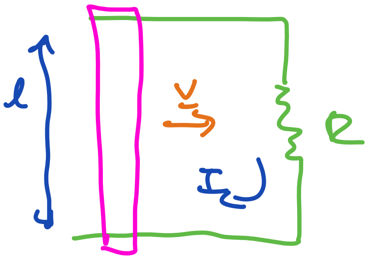
Thoughts on the sign of the flux rule

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

this gives the direction of the induced current

Lenz law: Nature abhors a change in flux
$\Leftrightarrow$
Induced current will oppose the change in flux of $\underline{B}$

Physical interpretation of Lenz law
$\times$ uniform B into the page

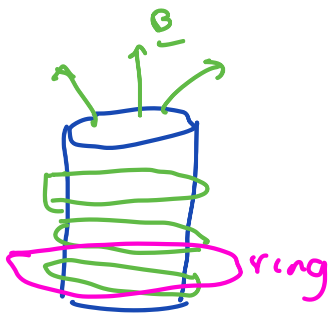


Lenz law: The magnetic force will try to slow down the motion of the bar.

The induced current will oppose the change in flux of B.

Lenz law tells us the direction of the induced current

Example



Before the current

$$\Phi_{ring} = 0$$

After we turn the current on there is $\underline{B}$ w/ flux upward

Change in flux generates emf → induced current in the ring.

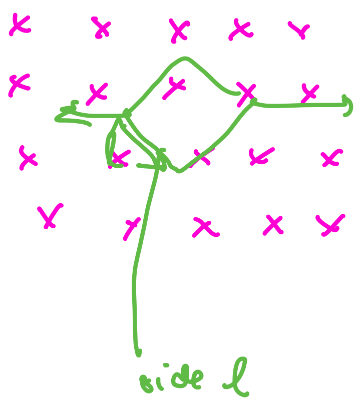
Direction of the induced current is such that it opposes change in flux

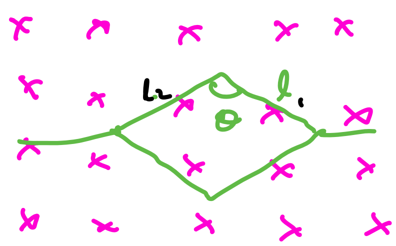
⇒ Current is opposite to the current in the solenoid.

What happens to opposite currents? they repel

Example: Loop changing in area

. A square loop with length ℓ is placed in a uniform $\underline{B}$ into the page. During a time interval Δt, the loop is pulled and turns into a rhombus. Assuming the resistance of the loop is R, find the average induced current & its direction.





We will use the flux rule to calculate the induced emf & get the current using Ohm's law

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\Delta \Phi}{\Delta t}$$

$$\Phi_B = \int \underline{B} \cdot d\underline{a} = B \int da = BA$$

$$\Rightarrow \mathcal{E} = -\frac{B \Delta A}{\Delta t}$$

The change in area $\Delta A = A_f - A_i$

$A_i = l^2$

$A_f = l^2 \sin\theta$ $\quad A = |\underline{l}_1 \times \underline{l}_2| = l_1 l_2 \sin\theta$ from geometrical interp of cross product

area of the parallelogram defined by vectors $\underline{l}_1$ and $\underline{l}_2$

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{l^2 \sin\theta - l^2}{\Delta t} = -\left[\frac{l^2 (1-\sin\theta)}{\Delta t}\right]$$

$$\therefore \ \varepsilon = B\left[\frac{l^2(1-\sin\theta)}{\Delta t}\right]$$

To determine the current we use Ohm's law.

$$\varepsilon = IR \qquad \Rightarrow I = \frac{B l^2 (1-\sin\theta)}{\Delta t R}$$

since $\frac{\Delta A}{\Delta t} < 0$ Flux through loop decreases

So the current flows clockwise to generate B into the page.

We can summarize this using the following conventions:

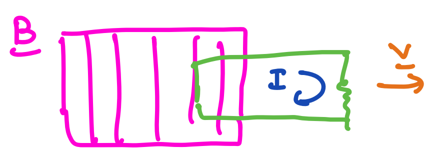
Φ_B	$\frac{d\Phi_B}{dt}$	ε	I
+	+	−	−
	−	+	+
−	+	−	−
	−	+	+

⊕ I counterclockwise

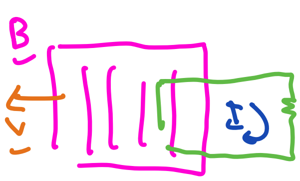
⊖ I clockwise

We define a positive direction for $\underline{A}$.

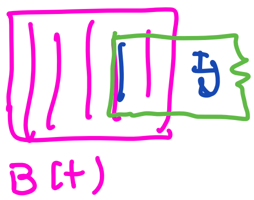
Faraday's law

① 

Loop moves to the right through $\underline{B}$ $\rightarrow$ induced current

② 

Pull magnet to left $\rightarrow$ induced current
Don't move loop

③  Changing $\underline{B}(t) \to$ induced current.

Experiment ① Motional emf drives current

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \underline{B}\cdot d\underline{a} = -\int \frac{\partial B}{\partial t}\cdot d\underline{a} \quad (*)$$

Experiment ② Same emf arises.

If the loop moves, $\to$ emf

but if it doesn't there is no emf $\to$ the force driving the current can't be magnetic

$$F_B = q(\underline{v} \times \underline{B})$$

for $\underline{v} = 0$ no magnetic force.

What exerts a F on stationary charges? F_E electric force

Faraday: $\boxed{\text{A changing magnetic field induces an electric field}}$

This induced electric field accounts for the emf that drives current in this case. And empirically:

$$\mathcal{E} = \oint \underline{E}\cdot d\underline{l} = \frac{d\Phi}{dt} \quad (**)$$

Equating the expressions for $\mathcal{E}$:

$$\oint \underline{E}\cdot d\underline{l} = -\int \frac{\partial \underline{B}}{\partial t}\cdot d\underline{a}$$ Faraday's law in integral form

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$ Faraday's law in differential form

Experiment #3:

Apply Faraday's law $\to$ we have a changing $B(t) \to$ induced $\underline{E}$, which gives to rise to an emf $-\frac{d\Phi}{dt}$ which drives current.

In general $\quad \mathcal{E} = -\frac{d\Phi}{dt}$

The induced electric field.

Faraday's is the generalization of $\nabla \times \underline{E} = 0$ in the case we have time-dependent quantities.

If $\underline{E}$ is due exclusively to a changing $\underline{B}$:

$$\nabla \cdot \underline{E} = 0 \quad , \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

in the case where we don't have static charge distributions $\rho = 0$

This looks like magnetostatics

$$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{B} = \mu_0 \underline{J}$$

We can write for $\underline{E}$ Ampère's law as we did for magnetostatics (change $-\frac{\partial \underline{B}}{\partial t}$ for $\mu_0 \underline{J}$) and we get:

$$\underline{E} = \frac{-1}{4\pi} \int \frac{(\partial \underline{B}/\partial t) \times \hat{r}}{r^2} d\tau = \frac{-1}{4\pi} \frac{\partial}{\partial t} \int \frac{\underline{B} \times \hat{r}}{r^2} d\tau$$

and Ampère's law

$$\oint \underline{E} \cdot d\underline{l} = -\frac{\partial \Phi}{\partial t}$$

This will be in the quasistatic regime. B changes slowly.

Example: A uniform $\underline{B}$ pointing straight up fills a region of space. If $\underline{B}(t)$ what is the induced $\underline{E}$?

We can calculate $\underline{E}$ using the equiv of Ampère's law?

$$\oint \underline{E} \cdot d l = -\frac{d\Phi}{dt}$$

$$\oint \underline{E} \cdot d\underline{l} = E(2\pi s)$$

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\left(B(t)\overbrace{\pi s^2}^{\text{area}}\right) = -\pi s^2 \frac{dB}{dt}$$

$$\Rightarrow E(2\pi s) = -\pi s^2 \frac{dB}{dt} \Rightarrow \underline{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\Phi}$$

$B(t)$ is the source of $\underline{E}$